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Lissajous Figures Revisited for Different Frequences

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Abstract . Lissajous figures are generated by two simple harmonic motions of a particle in two perpendicular directions with frequency/ period ratios (1:3) and (1:4) respectively. Equations of the two motions in either case in terms of displacements of the particle in two perpendicular directions are derived with the help of which the relevant lissajous figures are drawn.

1. Introduction

V. Satya Prakash[1] introduced lissajous figures associated with two frequency ratios (1:1) and (1:2) respectively for phase differences 0^0,90^0 and 180^0 and in general considering different amplitudes. Herein an attempt is made to project lissajous figures created by two different frequency ratios (1:3) and (1:4) respectively though the equations of motion with variables as the displacements along two perpendicular lines are complicated.

EQUATIONS OF CURVES IN LISSAJOUS FIGURES CREATED BY TWO PERPENDICULAR SIMPLE HARMONIC MOTIONS

Frequency ratio (1:3) with phase difference 0^0: Let us consider a two – dimensional system of axes XOY; OY is perpendicular to 0X, O being the origin and let a Particle move in such a way that its position(x,y) at any instant of time t is given by

$$x = a\sin t,\ y = b\sin 3t \tag{1}$$

where a and b are amplitudes.

Eliminating t between the above two displacement equations, we get recalling Trigonometry[2]

$$Y = b\,(3\sin t - \sin^3 t) = b\sin t(3 - 4\sin^2 t) = b\frac{x}{a}\left(3 - 4\frac{x^2}{a^2}\right) \tag{2}$$

For all lissajous curves ie for simple harmonic motion, obviously $x \leq a$ and $y \leq b$.In order to draw the curve(1) as a lissajous figure for 0^0 degree phase difference,we mainly see that the curve passes through the origin(0,0) and$(\pm\frac{\sqrt{3}a}{2},0)$. $\frac{dy}{dx}=\frac{b}{a}\left(3-\frac{12x^2}{a^2}\right)=0\; when\; x=\pm\frac{a}{2}$ and subsequently from (2) y=$\pm$b which suggests that the tangents to the curve at peak points$(\pm\frac{a}{2},\pm b)$ are parallel to the x-axis. Further x can increase through positive values beyond $\frac{\sqrt{3}a}{2}$, to yield negative values of y and can assume negative values beyond$-\frac{\sqrt{3}a}{2}$ to yield positive values of y. Positive values of x from 0 to $\frac{\sqrt{3}a}{2}$, give postive values of y and negative values of x from 0 to $\frac{-\sqrt{3}a}{2}$ give negative vales of y.Hence the curve passes through the points (0,0),$(\pm\frac{a}{2},\pm b)$, $(\pm a,\mp b)$, $(\pm\frac{\sqrt{3}a}{2},0)$, $\left(\pm\frac{a}{\sqrt{2}},\pm\frac{b}{\sqrt{2}}\right)$. Hence with all these inputs the curve depicting lissajous figure.

Now let us draw lissajous figure with the same frequency ratio(1:3) but with phase difference 90^0.Then the displacement equations of the two harmonic motions with amplitudes a and b are due to Trigonometry[2] :

$$x=a\sin t\;,\; y=b\sin(3t+90^0)=b\cos 3t \tag{3}$$

Or, $$y=b(4\cos^3 t-3\cos t)=b\cos t(4\cos^2 t-3)$$

Eliminating t between these two equations,

$$Y=\frac{b\sqrt{a^2-x^2}}{a^3}(a^2-4x^2) \tag{4}$$

which indicates that the lissajous curve passes through points $(\frac{\pm a}{2},0)$, $(\pm a,0)$, $\left(\pm\frac{a}{\sqrt{2}},-\frac{b}{\sqrt{2}}\right)$, $\left(\pm\frac{\sqrt{3}a}{2},-b\right)$ and $(0,b)$ and is symmetrical about Y-axis.

For frequency ratio(1:3)and phase difference 180^0, x= asint and y=b sin(3t+180^0)= $-$bsin3t

so that $$y=b\frac{x}{a}\left(4\frac{x^2}{a^2}-3\right) \tag{5}$$

(x>0, $\frac{4x^2-3a^2}{a^2}$>0) and (x<0, $\frac{4x^2-3a^2}{a^2}$<0) imply positive values of y while

(x<0, $\frac{4x^2-3a^2}{a^2}$>0) and (x>0, $\frac{4x^2-3a^2}{a^2}$<0) imply negative values of y.

which entail that this lissajous curve passes through the Origin(0,0) , $(\pm\frac{\sqrt{3}a}{2},0)$, $(\pm\frac{a}{2}.\mp b)$,$\left(\pm\frac{a}{\sqrt{2}},\mp\frac{b}{\sqrt{2}}\right)$, $(\pm a,\pm b)$.Therefore the relevant curve.

Lissajous figure for frequency ratio (1:3) with phase difference 270^0 :

The displacement equations are

$$x=a\sin t, \quad y=b\sin(3t+270^0)=-b\cos 3t \tag{6}$$

The curve passes through the points points

$\left(\frac{\pm a}{2},0\right), (\pm a,0), \left(\pm\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right), \left(\pm\frac{\sqrt{3}a}{2},b\right)$ *and* $(0,-b)$ and is symmetrical about Y-axis.

Lissajous figure for frequency ratio (1:4) with phase difference 0^0:

The displacement equations are

$$x=a\sin t \text{ and } y=b\sin 4t \tag{7}$$

Eliminating t between these equations the equation of the curve is obtained as

$$Y=4\frac{bx}{a^3}(a^2-2x^2)\sqrt{(a^2-x^2)} \tag{8}$$

Passing through $(\pm a,0), \left(\pm\frac{a}{\sqrt{2}},0\right)$, $(0,0), \left(\pm\frac{a}{2},\pm\frac{\sqrt{3}b}{2}\right)$, $\left(\pm\frac{\sqrt{3}a}{2},\mp\frac{\sqrt{3}b}{2}\right)$, $\left(\pm a\sin\frac{\pi}{8}.\pm b\right), \left(\pm a\sin 3\frac{\pi}{8},\mp b\right)$. $\sin\frac{\pi}{8}=.38$, $\sin 3\frac{\pi}{8}=.92$.

Lissajous figure with frequency ratio (1:4) and phase difference 90^0:

In this case the displacement equations become by use of Trigonometry[2]

$x=a\sin t, y=b\sin(4t+90^0)=b\cos 4t = b(2\cos^2 2t-1)=b[2\{1-(2\sin^2 t)\}^2-1]$ which leads to the equation of the curve

$$y=b[2(1-2x^2)^2-1] \tag{9}$$

Hence the curve (8) passes through

$(\pm a,b), \left(\pm a\sin\frac{\pi}{8},0\right), \left(\pm a\sin 3\frac{\pi}{8},0\right), \left(\pm\frac{a}{\sqrt{2}},-b\right), \left(\pm\frac{a}{2},-\frac{b}{2}\right), \left(\pm\frac{\sqrt{3}a}{2},-\frac{b}{2}\right)$.

$\sin\frac{\pi}{8}=.38, \sin 3\frac{\pi}{8}=.92.$ The curve is symmetrical about $Y-$axis.

Lissajous figure with frequency ratio (1:4) and phase difference 180^0:

Here the displacement equations can be written as

$$x=a\sin t, \quad y=b\sin(4t+180^0)=-b\sin 4t \tag{10}$$

which give

$$Y=-4\frac{bx}{a^3}(a^2-2x^{2)}\sqrt{(a^2-x^2)} \tag{11}$$

Hence the curve (10) passes through

$(\pm a,0), \left(\pm\frac{a}{\sqrt{2}},0\right)$, origin$(0,0)$, $\left(\pm\frac{a}{2},\mp\frac{\sqrt{3}b}{2}\right), \left(\pm\frac{\sqrt{3}a}{2},\pm\frac{\sqrt{3}b}{2}\right)$,

$\left(\pm a\sin\frac{\pi}{8},\mp b\right), \left(\pm a\sin\frac{3\pi}{8},\pm b\right)$.

Lissajous figure with frequency (1:4) and phase difference 270^0 :

Then by Trigonometry[2] the displacement equations are

X=asint and y=bcos(4t+270^0)=−bcos4t

In this case the curve passes through

$(\pm a, -b),(\pm a\sin\frac{\pi}{8}, 0)$, $(\pm a\sin 3\frac{\pi}{8}, 0)$, $\left(\pm\frac{a}{\sqrt{2}}, b\right), \left(\pm\frac{a}{2},\frac{b}{2}\right), (\pm\frac{\sqrt{3}a}{2},\frac{b}{2}),(0,-b)$.

Using these points the curve is drawn, which is symmetrical about Y-axis.

References

[1] V Satya Prakash, Using Sound Card Oscilloscope For Generation of Lissajous Figures, *IAPT Bulletin*, **11(12)**, .344 (2019)

[2] H S Hall, *Text Book of Trigonometry*, *Circular Functions of Angles*, 96 (1905)